Scientific Insights

Article

Bilateral Quadratic Series Via Residue Method

Xiaojing Chen [1] and Wenchang Chu [2,*]

[1] School of Statistics and Data Science, Qufu Normal University, Qufu 273165, China
[2] Independent Researcher, Via Dalmazio Birago 9/E, Lecce 73100, Italy
* Correspondence: hypergeometricx@outlook.com

Received: 16 December 2024; Accepted: 3 April 2025; Published: 27 April 2025

Abstract: The bilateral quadratic $_7H_7$-series is examined and evaluated by making use of the Cauchy residue method. The main theorem presents an analytic formula for this series, which contains five closed summation formulae as consequences. Just like the classical hypergeometric series, these remarkable formulae should find potential applications in pure mathematics and theoretical physics.

Keywords: classical hypergeometric series; bilateral quadratic series; the gamma function; meromorphic function; the Cauchy residue method

1. Introduction and Motivation

Let $\mathbb{Z}$ and $\mathbb{N}$ be the sets of integers and natural numbers with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then for an indeterminate x and $n \in \mathbb{Z}$, the shifted factorial can be expressed as the quotient

$$(x)_n = \Gamma(x+n)/\Gamma(x).$$

Here the Γ-function is defined by Euler integral:

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\mathrm{d}u, \text{ with } \Re(x) > 0,$$

which satisfies Euler's reflection formula (cf. Rainville [1, §17])

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \tag{1}$$

and Legendre's duplication formula (cf. Rainville [1, §19])

$$\Gamma(2x) = \frac{2^{2x}}{2\sqrt{\pi}}\Gamma(x)\Gamma(x+\frac{1}{2}). \tag{2}$$

Throughout the paper, we shall adopt the following notation of Bailey [2] and Slater [3] for the generalized hypergeometric series

$$_{1+p}F_p\left[\begin{matrix}a_0, a_1, \cdots, a_p\\ b_1, \cdots, b_p\end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k\cdots(a_p)_k}{k!\,(b_1)_k\cdots(b_p)_k} z^k,$$

$$_pH_p\left[\begin{matrix}a_1, \cdots, a_p\\ b_1, \cdots, b_p\end{matrix}; z\right] = \sum_{k=-\infty}^{\infty} \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_p)_k} z^k.$$

There exist numerous hypergeometric series identities in the literature (see Bailey [2], Slater [3], Brychkov [4, Chapter 8] and [5–11], just for example). However, for bilateral hypergeometric series (cf. Slater [3, Chapter 6]), there are relatively few closed formulae:

- Dougall [12, 1907]: $\Re(c+d-a-b) > 1$

$$ {}_2H_2\left[\begin{matrix}a,b\\c,d\end{matrix};1\right]=\Gamma\left[\begin{matrix}1-a,1-b,c,d,c+d-a-b-1\\c-a,d-a,c-b,d-b\end{matrix}\right]. $$

- M. Jackson [13, Equation 2.3]: $\Re(2+b+d-a-c)>0$

$$ {}_3H_3\left[\begin{matrix}a,c,\frac{1+b+d}{2}\\1+b,1+d,\frac{1+a+c}{2}\end{matrix};1\right]=2^{a+c-b-d-1}\pi\left\{\frac{cos\ \frac{a-c}{2}\pi}{cos\ \frac{a+c}{2}\pi}+\frac{cos\ \frac{b-d}{2}\pi}{cos\ \frac{b+d}{2}\pi}\right\} $$
$$ \times\left[\begin{matrix}1-\mathrm{a},1-\mathrm{c},1+\mathrm{b},1+\mathrm{d},\frac{2-\mathrm{a}-\mathrm{c}+\mathrm{b}+\mathrm{d}}{2}\\\frac{1-\mathrm{a}-\mathrm{c}}{2},\frac{1+\mathrm{b}+\mathrm{d}}{2},\frac{2-\mathrm{a}+\mathrm{b}}{2},\frac{2-\mathrm{a}+\mathrm{d}}{2},\frac{2-\mathrm{c}+\mathrm{b}}{2},\frac{2-\mathrm{c}+\mathrm{d}}{2}\end{matrix}\right]. $$

- Dougall [12, 1907]: $\Re(1+2a-b-c-d-e)>0$

$$ {}_5H_5\left[\begin{matrix}1+\frac{a}{2},\quad b,\quad c,\quad d,\quad e\\\frac{a}{2},1+a-b,1+a-c,1+a-d,1+a-e\end{matrix};1\right] $$
$$ =\Gamma\left[\begin{matrix}1+a-b,1+a-c,1+a-d,1+a-e,1-b,1-c,1-d,1-e,1+2a-b-c-d-e\\1+a,1-a,1+a-b-c,1+a-b-d,1+a-b-e,1+a-c-d,1+a-c-e,1+a-d-e\end{matrix}\right]. $$

- M. Jackson [14, Equation 1.2]: $\Re(2c-a)>0$

$$ {}_6H_6\left[\begin{matrix}1+\frac{a}{2}, & b,c,d, & 1+2a-b-2c,1+2a-2c-d\\\frac{a}{2}, & 1+a-b,1+a-c,1+a-d, & b-a+2c,d-a+2c\end{matrix};-1\right] $$
$$ =\Gamma\left[\begin{matrix}1+a-b,1+a-d,1-b,1-d,b-a+2c,d-a+2c,b-2a+2c,d-2a+2c\\1+a,1-a,c,c-a,2c-a,1+a-b-d,b+d-3a+4c\end{matrix}\right] $$
$$ \times\Gamma\left[\begin{matrix}\frac{1+a-b-d}{2},\frac{1-3a+b+d}{2}+2c\\\frac{1-a+b-d}{2}+c,\frac{1-a-b+d}{2}+c\end{matrix}\right]\left\{1+\sin\,\pi\left[\begin{matrix}a-b-c,a-c-d\\c,c-a\end{matrix}\right]\right\}. $$

These formulae can be proved by the modified Abel lemma on summation by parts [15] and the Cauchy residue method [16].

In this paper, we aim at evaluating the following bilateral quadratic ${}_7H_7$-series

$$ \mathcal{H}(a,b,c,d,e)=\sum_k\frac{2a+3k}{2a}\left[\begin{matrix}b, & d, & \frac{1}{2}+2a-b-d\\1+2a-2b, & 1+2a-2d, & 2b+2d-2a\end{matrix}\right]_k \times\left[\begin{matrix}2c, & 2e, & 1+2a-2c-2e\\1+a-c, & 1+a-e, & \frac{1}{2}+c+e\end{matrix}\right]_k. \tag{3} $$

This will add a significantly new series with its analytic expression to the existing few bilateral summation formulae just displayed.

The $\mathcal{H}$-series is convergent because the sum of its denominator parameters exceeds by 2 that of its numerator ones. In order to examine the above $\mathcal{H}$-series, we shall introduce, in the next section, the meromorphic Ω-function and express residue sums over seven classes of its poles in terms of three symmetric functions with respect to $\{b, d\}$ and to $\{c, e\}$. Then in the third section, we shall establish the general summation theorem and derive consequently five closed formulae with three for bilateral series and two for unilateral ones.

For the sake of brevity, the multi–parameter forms of the Γ-function and the trigonometric sine function will be abbreviated respectively to

$$ \Gamma\left[\begin{matrix}\alpha,\beta,\cdots,\gamma\\A,B,\cdots,C\end{matrix}\right]=\frac{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\gamma)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}, $$
$$ \sin\,\pi\left[\begin{matrix}\alpha,\beta,\cdots,\gamma\\A,B,\cdots,C\end{matrix}\right]=\frac{\sin\,(\pi\alpha)\sin\,(\pi\beta)\cdots\sin\,(\pi\gamma)}{\sin\,(\pi A)\sin\,(\pi B)\cdots\sin\,(\pi C)}. $$

Throughout the paper, several cumbersome expressions are simplified by appropriately devised "Mathematica" (Wolfram, Version 11) commands, without reproducing their tedious details. Furthermore, most displayed equations are experimentally checked in order to assure the accuracy.

2. The Cauchy Residue Method

Define the meromorphic function by

$$\Omega(z)=\frac{\pi(2a+3z)}{\tan(\pi z)}\Gamma\left[\begin{matrix}2c+z,2e+z,1+2a-2c-2e+z\\1+a-c+z,1+a-e+z,\frac{1}{2}+c+e+z\end{matrix}\right]$$
$$\times\Gamma\left[\begin{matrix}b+z,d+z,\frac{1}{2}+2a-b-d+z\\1+2a-2b+z,1+2a-2d+z,2b+2d-2a+z\end{matrix}\right],$$

where for the Γ-function quotient, the sum of its denominator parameters exceeds that of its numerator parameters by 3.

It is not hard to check that all of the singular points of $\Omega(z)$ are simple poles, which can be divided into seven classes:

$$\mathrm{AA}=\{z=n\mid n\in\mathbb{Z}\},$$
$$\mathrm{BB}=\{z=-b-n\mid n\in\mathbb{N}_0\},$$
$$\mathrm{DD}=\{z=-d-n\mid n\in\mathbb{N}_0\},$$
$$\mathrm{BD}=\left\{z=b+d-2a-\frac{1}{2}-n\middle|n\in\mathbb{N}_0\right\},$$
$$\mathrm{CC}=\{z=-2c-n\mid n\in\mathbb{N}_0\},$$
$$\mathrm{EE}=\{z=-2e-n\mid n\in\mathbb{N}_0\},$$
$$\mathrm{CE}=\{z=2c+2e-2a-1-n\mid n\in\mathbb{N}_0\}.$$

Let $\mathcal{C}_n(\varepsilon)$ be the circle of radius $n+\varepsilon$ centered at the origin with the $\varepsilon>0$ being chosen such that $\mathcal{C}_n(\varepsilon)$ does not pass any poles of $\Omega(z)$. In view of the Stirling formula (cf. Rainville [1, §22]), there holds, for $z=x+yi\in\mathcal{C}_n(\varepsilon)$ with $x\geq 0$ (i.e., z lies on the right half circle), the following asymptotic relation:

$$|\Omega(z)\tan(\pi z)|\approx\mathcal{O}\left(\frac{1}{|z|^2}\right)\quad\text{as } |z|\to\infty. \tag{4}$$

This relation is also valid when $z=x+yi$ lies on the left half of the circle $\mathcal{C}_n(\varepsilon)$ with $x\leq 0$. In fact, by making use of the reflection formula (1), we can express

$$\Omega(z)=\omega(z)\times\vartheta(z),$$

where

$$\omega(z)=\frac{\pi(2a+3z)}{\tan(\pi z)}\Gamma\left[\begin{matrix}c-a-z,e-a-z,\frac{1}{2}-c-e-z\\1-2c-z,1-2e-z,2c+2e-2a-z\end{matrix}\right]$$
$$\times\Gamma\left[\begin{matrix}2b-2a-z,2d-2a-z,1+2a-2b-2d-z\\1-b-z,1-d-z,\frac{1}{2}-2a+b+d-z\end{matrix}\right]$$

and

$$\vartheta(z)=\sin\pi\left[\begin{matrix}2a-2b+z,2a-2d+z,2b+2d-2a+z\\b+z,d+z,2a-b-d-\frac{1}{2}+z\end{matrix}\right]$$
$$\times\sin\pi\left[\begin{matrix}a-c+z,a-e+z,\frac{1}{2}+c+e+z\\2c+z,2e+z,2a-2c-2e+z\end{matrix}\right].$$

Now for $z=x+yi\in\mathcal{C}_n(\varepsilon)$ with $x<0$, we have the asymptotic relation

$$|\omega(z)\tan(\pi z)|\approx\mathcal{O}\left(\frac{1}{|z|^2}\right)\quad\text{as } |z|\to\infty \tag{5}$$

thanks again to the Stirling formula.

For the trigonometric fraction $\vartheta(z)$, its factors in the numerator can be paired off with those in the denominator so that $\vartheta(z)$ is written as a product of six fractions of the following form $\frac{\sin\pi(\xi+z)}{\sin\pi(\eta+z)}$ with ξ and η being fixed complex numbers subject to $\eta+z\notin\mathbb{Z}$.

Writing further $\xi=\sigma+\rho i,\eta=\tau+\lambda i$ and $z=x+yi\in\mathcal{C}_n(\varepsilon)$, we can check without difficulty that as $n\to\infty$

$$\begin{aligned}\left|\frac{\sin\pi(\xi+z)}{\sin\pi(\eta+z)}\right|^2 &= \frac{\sin\pi(\xi+z)}{\sin\pi(\eta+z)}\frac{\sin\pi(\overline{\xi}+\overline{z})}{\sin\pi(\overline{\eta}+\overline{z})}\\ &= \frac{\cos 2\pi(\rho+y)i-\cos 2\pi(\sigma+x)}{\cos 2\pi(\lambda+y)i-\cos 2\pi(\tau+x)}=\mathcal{O}(1)\end{aligned}$$

which implies that $|\vartheta(z)|=\mathcal{O}(1)$. Hence, the asymptotic relation (4) is validated also for $z=x+yi\in\mathcal{C}_n(\varepsilon)$ with $x<0$.

When $n\to\infty$, there holds analogously for any $z=x+yi\in\mathcal{C}_n(\varepsilon)$:

$$|\tan(\pi z)|^{-2}=|\cot(\pi z)|^2=\cot\pi z\cot\pi\overline{z}=\frac{\cos 2\pi yi+\cos 2\pi x}{\cos 2\pi yi-\cos 2\pi x}=\mathcal{O}(1).$$

Therefore for sufficient large n, the following inequality holds

$$\left|\frac{1}{2\pi i}\oint_{\mathcal{C}_n(\varepsilon)}\Omega(z)\mathrm{d}z\right|\le\mathcal{O}\left\{(n+\varepsilon)\max_{z\in\mathcal{C}_n(\varepsilon)}|\Omega(z)|\right\}\le\mathcal{O}\left\{\frac{1}{n+\varepsilon}\right\}.$$

Consequently, we have the limiting relation:

$$\frac{1}{2\pi i}\oint_{\mathcal{C}_n(\varepsilon)}\Omega(z)\ \mathrm{d}z\to 0\ \text{ as } n\to\infty. \tag{6}$$

According to the Cauchy residue theorem (cf. Titchmarsh [17, §3.11]), the sum of the residues of $\Omega(z)$ at its singular points inside $\mathcal{C}_n(\varepsilon)$ is equal to zero. Denote by $\Re(\mathrm{S})$ the residue sum of $\Omega(z)$ at its poles in S. Then we have the equality

$$\Re(\mathrm{AA})=-\Re(\mathrm{BB})-\Re(\mathrm{DD})-\Re(\mathrm{BD})-\Re(\mathrm{CC})-\Re(\mathrm{EE})-\Re(\mathrm{CE}). \tag{7}$$

First of all, it is quite routine to write down that

$$\begin{aligned}\Re(\mathrm{AA}) &= \sum_{n=-\infty}^{\infty}\mathrm{Res}_{z=n}\{\Omega(z)\}=2a\mathcal{H}(a,b,c,d,e)\\ &\times\Gamma\left[\begin{matrix}b,d,\frac{1}{2}+2a-b-d,2c,2e,1+2a-2c-2e\\ 1+2a-2b,1+2a-2d,2b+2d-2a,1+a-c,1+a-e,\frac{1}{2}+c+e\end{matrix}\right].\end{aligned}$$

In order to find a closed formula for the $\mathcal{H}$-series, we have to evaluate the residue sums for the remaining classes of poles. Fortunately, these six residue sums can be expressed, in synchronization, as the following three symmetric functions in $\{b,d\}$ and in $\{c,e\}$:

$$\begin{aligned}&\mathcal{F}(a,b,c,d,e)\\ &= {}_3F_2\left[\begin{matrix}1+2a-b-c-d-e,\frac{1}{2}+b-c-e,\frac{1}{2}-c+d-e\\ \frac{3}{2}+a-2c-e,\frac{3}{2}+a-c-2e\end{matrix};1\right],\end{aligned} \tag{8}$$

$$\begin{aligned}&\mathcal{G}_\phi(a,b,c,d,e)\\ &=\Gamma\left[\begin{matrix}1+2a-b-c-d-e,\frac{1}{2}+b-c-e,\frac{1}{2}+d-c-e,\frac{1}{2}+a+c-b-d,\frac{1}{2}+a+e-b-d\\ 1+a-b-c,1+a-b-e,1+a-c-d,1+a-d-e\end{matrix}\right],\end{aligned} \tag{9}$$

$$\mathcal{G}_{\psi}(a,b,c,d,e) = \Gamma\begin{bmatrix} 1+2a-b-c-d-e, \frac{1}{2}+b-c-e, \frac{1}{2}-c+d-e \\ \frac{3}{2}+a-2c-e, \frac{3}{2}+a-c-2e \end{bmatrix}. \tag{10}$$

2.1. Residue Sums over BB and DD.

For the unilateral quadratic series, there exists the following transformation.

Lemma 1 (Rahman [18, Equation 6.4]).

$$\begin{aligned} &{}_7F_6\begin{bmatrix} 1+\frac{2a}{3}, 2a, 2c, 1-2c, \quad b, d, \frac{1}{2}+2a-b-d \\ \frac{2a}{3}, \frac{1}{2}+a+c, 1+a-c, \quad 1+2a-2b, 1+2a-2d, 2b+2d-2a \end{bmatrix}; 1 \\ &= \Gamma\begin{bmatrix} 1+2a-2b, 1+2a-2d, \frac{1}{2}+a+c, 1+a-c, \frac{1}{2}+a-b+c-d, 1+a-b-c-d \\ 1+2a, 1+2a-2b-2d, \frac{1}{2}+a-b+c, \frac{1}{2}+a+c-d, 1+a-b-c, 1+a-c-d \end{bmatrix} \\ &+\Gamma\begin{bmatrix} 1+2a-2b, 1+2a-2d, 2b+2d-2a, 1+a-c, \frac{1}{2}+a+c, \frac{1}{2}+a-b+c-d \\ 1+2a, b, d, \frac{3}{2}+2a-2b-d, \frac{3}{2}+2a-b-2d \end{bmatrix} \\ &\times \Gamma\begin{bmatrix} 1+a-b-c-d \\ 2c, 1-2c \end{bmatrix} {}_3F_2\begin{bmatrix} 1+a-b-c-d, \frac{1}{2}+2a-b-d, \frac{1}{2}+a-b+c-d \\ \frac{3}{2}+2a-2b-d, \frac{3}{2}+2a-b-2d \end{bmatrix}; 1 \end{aligned}.$$

Evaluating the residue sum over BB

$$\begin{aligned} \Re(\mathrm{BB}) = \sum_{n=0}^{\infty} \mathrm{Res}_{z=-b-n}\{\Omega(z)\} &= \frac{\sin\,\pi(2a-b-2d)}{\tan\,\pi b} \\ &\times \Gamma\begin{bmatrix} d-b, \frac{1}{2}+2a-2b-d, 2c-b, 2e-b, 1+2a-b-2c-2e \\ 2a-3b, 1+a-b-c, 1+a-b-e, \frac{1}{2}-b+c+e \end{bmatrix} \\ \times \quad {}_7F_6&\begin{bmatrix} 3b-2a, 1+\frac{3b-2a}{3}, b-2a+2d, 1+2a-b-2d, b+c-a, b+e-a, \frac{1}{2}+b-c-e \\ \frac{3b-2a}{3}, 1+b-d, \frac{1}{2}-2a+2b+d, 1+b-2c, 1+b-2e, b-2a+2c+2e \end{bmatrix}; 1 \end{aligned}$$

and then reformulating the above ${}_7F_6$-series by Lemma 1, we get, after reordering the terms, the following explicit formula

$$\Re(\mathrm{BB})=\frac{\sin\pi(2a-b-2d)}{\tan\pi b}\Gamma\begin{bmatrix}\frac{1}{2}-2a+2b+d,\frac{1}{2}+2a-2b-d\\ \frac{1}{2}-a+b-c+d,\frac{1}{2}-a+b+d-e\end{bmatrix}$$

$$\times\Gamma\begin{bmatrix}d-b,1+b-d,2c-b,1+b-2c,2e-b,1+b-2e\\ 2a-3b,1-2a+3b,\frac{1}{2}+b-c-e,\frac{1}{2}-b+c+e\end{bmatrix}$$

$$\times\Gamma\begin{bmatrix}1+2a-b-c-d-e,\frac{1}{2}+b-c-e,\frac{1}{2}-c+d-e\\ 1+a-b-c,1+a-b-e,1+a-c-d,1+a-d-e\end{bmatrix}$$

$$+\frac{\sin\pi(2a-b-2d)}{\tan\pi b}\Gamma\begin{bmatrix}1+2a-b-c-d-e,\frac{1}{2}+b-c-e,\frac{1}{2}-c+d-e\\ \frac{3}{2}+a-2c-e,\frac{3}{2}+a-c-2e\end{bmatrix}$$

$$\times\Gamma\begin{bmatrix}d-b,1+b-d,1+b-2e,b-2a+2c+2e,1+2a-b-2c-2e\\ 2a-3b,1-2a+3b,b+c-a,1+a-b-c,b+e-a,1+a-b-e\end{bmatrix}$$

$$\times\Gamma\begin{bmatrix}2c-b,1+b-2c,2e-b,\frac{1}{2}+2a-2b-d,\frac{1}{2}-2a+2b+d\\ b-2a+2d,1+2a-b-2d,\frac{1}{2}+b-c-e,\frac{1}{2}-b+c+e\end{bmatrix}$$

$$\times\ {}_3F_2\begin{bmatrix}1+2a-b-c-d-e,\frac{1}{2}+b-c-e,\frac{1}{2}-c+d-e\\ \frac{3}{2}+a-2c-e,\frac{3}{2}+a-c-2e\end{bmatrix};1\Bigg].$$

In terms of the three symmetric functions defined by (8), (9) and (10), we can write shortly the residue sum as

$$\Re(\mathrm{BB})=\mathcal{G}_\phi(a,b,c,d,e)\phi_1(a,b,c,d,e)+\mathcal{G}_\psi(a,b,c,d,e)\psi_1(a,b,c,d,e)\mathcal{F}(a,b,c,d,e)$$

where ϕ_1 and ψ_1 are quotients of trigonometric functions

$$\phi_1(a,b,c,d,e)=\sin\pi\begin{bmatrix}b+2d-2a,2a-3b,\frac{1}{2}+b,\frac{1}{2}+b-c-e,\frac{1}{2}+a-b+c-d,\frac{1}{2}+a-b-d+e\\ b,b-d,b-2c,b-2e,\frac{1}{2}+2a-2b-d\end{bmatrix},$$

$$\psi_1(a,b,c,d,e)=\sin\pi\begin{bmatrix}2a-b-2d,2a-b-2d,3b-2a,a-b-c,a-b-e,\frac{1}{2}+b,\frac{1}{2}+b-c-e\\ b,b-d,b-2c,b-2e,2a-b-2c-2e,\frac{1}{2}+2a-2b-d\end{bmatrix}.$$

Because the function $\Omega(z)$ is symmetric with respect to b and d, we have directly

$$\Re(\mathrm{DD})=\mathcal{G}_\phi(a,b,c,d,e)\phi_1(a,d,c,b,e)+\mathcal{G}_\psi(a,b,c,d,e)\psi_1(a,d,c,b,e)\mathcal{F}(a,b,c,d,e).$$

2.2. Residue Sum over BD

Next, evaluating the residue sum

$$\Re(\mathrm{BD})=\sum_{n=0}^{\infty}\mathrm{Res}_{z=b+d-2a-\frac{1}{2}-n}\{\Omega(z)\}=\frac{-\pi}{\tan\pi(\frac{1}{2}+2a-b-d)}$$

$$\times\Gamma\left[\begin{matrix}2b+d-2a-\frac{1}{2},b+2d-2a-\frac{1}{2},b+2c+d-2a-\frac{1}{2},b+d+2e-2a-\frac{1}{2},b-2c+d-2e+\frac{1}{2}\\ \frac{1}{2}-b+d,\frac{1}{2}+b-d,3b+3d-4a-\frac{3}{2},\frac{1}{2}-a+b-c+d,\frac{1}{2}-a+b+d-e,b+c+d+e-2a\end{matrix}\right]$$

$$\times\ {}_7F_6\left[\begin{matrix}\frac{3}{2}+4a-3b-3d,\frac{3}{2}+\frac{4a}{3}-b-d,\frac{1}{2}+b-d,\frac{1}{2}-b+d\\ \frac{1}{2}+\frac{4a}{3}-b-d,\frac{3}{2}+2a-2b-d,\frac{3}{2}+2a-b-2d\end{matrix}\right.\vdots$$

$$\left.\vdots\begin{matrix}\frac{1}{2}+a-b+c-d,\frac{1}{2}+a-b-d+e,1+2a-b-c-d-e\\ \frac{3}{2}+2a-b-2c-d,\frac{3}{2}+2a-b-d-2e,\frac{1}{2}-b+2c-d+2e\end{matrix};1\right]$$

and then transforming this ${}_7F_6$-series by Lemma 1, we derive, after simplifying the resulting expression, the following explicit formula

$$\Re(\mathrm{BD})=\mathcal{G}_{\phi}(a,b,c,d,e)\phi_2(a,b,c,d,e)+\mathcal{G}_{\psi}(a,b,c,d,e)\psi_2(a,b,c,d,e)\mathcal{F}(a,b,c,d,e)$$

where ϕ_2 and ψ_2 are quotients of trigonometric functions

$$\phi_2(a,b,c,d,e)=\sin\pi\left[\begin{matrix}\frac{1}{2}+4a-3b-3d,\frac{1}{2}+a-b+c-d,\frac{1}{2}+a-b-d+e\\ \frac{1}{2}+2a-b-d,\frac{1}{2}+2a-2b-d,\frac{1}{2}+2a-b-2d\end{matrix}\right]$$
$$\times\sin\pi\left[\begin{matrix}2a-b-c-d-e,b+d-2a,\frac{1}{2}+b-d\\ \frac{1}{2}+2a-b-2c-d,\frac{1}{2}+2a-b-d-2e\end{matrix}\right],$$

$$\psi_2(a,b,c,d,e)=\sin\pi\left[\begin{matrix}2a-b-c-d-e,b+d-2a,\frac{1}{2}+4a-3b-3d\\ \frac{1}{2}+2a-b-d,\frac{1}{2}+2a-b-2c-d,\frac{1}{2}+2a-b-d-2e\end{matrix}\right]$$
$$\times\sin\pi\left[\begin{matrix}\frac{1}{2}+b-d,\frac{1}{2}+b-d,\frac{1}{2}+a-b+c-d,\frac{1}{2}+a-b-d+e\\ \frac{1}{2}+b-2c+d-2e,\frac{1}{2}+2a-b-2d,\frac{1}{2}+2a-2b-d\end{matrix}\right].$$

2.3. Residue sums over CC and EE

For the unilateral quadratic series, there is another shorter transformation.

Lemma 2 (Rahman [18, Equation 6.5]).

$${}_7F_6\left[\begin{matrix}a,1+\frac{2a}{3}, & b,\frac{1}{2}+a-b, & 2c,2e,1+2a-2c-2e\\ \frac{2a}{3}, & 2b,1+2a-2b, & 1+a-c,1+a-e,\frac{1}{2}+c+e\end{matrix};1\right]$$
$$=\Gamma\left[\begin{matrix}\frac{1}{2},1+a-c,1+a-e,\frac{1}{2}+c+e\\ 1+a,\frac{1}{2}+c,\frac{1}{2}+e,1+a-c-e\end{matrix}\right]\ {}_3F_2\left[\begin{matrix}c,e,\frac{1}{2}+a-c-e\\ 1+a-b,\frac{1}{2}+b\end{matrix};1\right].$$

Writing down the residue sum over CC

$$\Re(\mathrm{CC})=\sum_{n=0}^{\infty}\operatorname{Res}_{z=-2c-n}\{\Omega(z)\}=\frac{-2\pi}{\tan(2\pi c)}$$
$$\times\Gamma\left[\begin{matrix}b-2c,d-2c,2e-2c,1+2a-4c-2e,\frac{1}{2}+2a-b-2c-d\\1+2a-2b-2c,1+2a-2c-2d,1+a-2c-e,a-3c,2b+2d-2a-2c,\frac{1}{2}-c+e\end{matrix}\right]$$
$$\times{}_7F_6\left[\begin{matrix}1+2c-\frac{2a}{3},3c-a,\frac{1}{2}+c-e,2c-a+e,\quad 2b+2c-2a,2c+2d-2a,1+2a-2b+2c-2d\\2c-\frac{2a}{3},1+2c-2e,4c-2a+2e,\quad 1+2c-b,1+2c-d,\frac{1}{2}-2a+b+2c+d\end{matrix};1\right]$$

and reformulating the above ${}_7F_6$-series by Lemma 2, we have alternatively

$$\Re(\mathrm{CC})=\frac{-2\pi}{\tan(2\pi c)}\Gamma\left[\begin{matrix}b-2c,1+2c-b,d-2c,1+2c-d,\frac{1}{2}+2a-b-2c-d,\frac{1}{2}-2a+b+2c+d\\a-3c,1+3c-a,1+2a-2b-2c,1+2a-2c-2d,\frac{1}{2}-a+b+c,\frac{1}{2}-a+c+d\end{matrix}\right]$$
$$\times\Gamma\left[\begin{matrix}\frac{1}{2},1+2a-4c-2e,2e-2c\\1+a-2c-e,\frac{1}{2}-c+e,1+a-b+c-d,2b+2d-2a-2c\end{matrix}\right]$$
$$\times{}_3F_2\left[\begin{matrix}b+c-a,c+d-a,\frac{1}{2}+a-b+c-d\\1+c-e,\frac{1}{2}-a+2c+e\end{matrix};1\right].$$

The last ${}_3F_2$-series can further be expressed as

$${}_3F_2\left[\begin{matrix}b+c-a,c+d-a,\frac{1}{2}+a-b+c-d\\1+c-e,\frac{1}{2}-a+2c+e\end{matrix};1\right]$$
$$=\left[\begin{matrix}1+c-e,1+2a-b-c-d-e,\frac{1}{2}+b-c-e,\frac{1}{2}+d-c-e\\\frac{1}{2}-a+b+d-e,1+a-b-e,1+a-d-e,\frac{1}{2}+a-2c-e\end{matrix}\right]$$
$$-\mathcal{F}(a,b,c,d,e)\times\Gamma\left[\begin{matrix}1+2a-b-c-d-e,1+c-e\\b+c-a,d+c-a\end{matrix}\right]$$
$$\times\Gamma\left[\begin{matrix}2c+e-a-\frac{1}{2},\frac{1}{2}+b-c-e,\frac{1}{2}+d-c-e\\\frac{1}{2}+a-b+c-d,\frac{3}{2}+a-c-2e,\frac{1}{2}+a-2c-e\end{matrix}\right]$$

thanks to the nonterminating Saalschützian summation formula

$${}_3F_2\left[\begin{matrix}a,c,e\\b,d\end{matrix}\middle|1\right]=\Gamma\left[\begin{matrix}b,1+a-d,1+c-d,1+e-d\\b-a,b-c,b-e,1-d\end{matrix}\right]$$
$$-\Gamma\left[\begin{matrix}b,1+a-d,1+c-d,1+e-d,d-1\\a,c,e,1+b-d,1-d\end{matrix}\right]{}_3F_2\left[\begin{matrix}1+a-d,1+c-d,1+e-d\\1+b-d,2-d\end{matrix};1\right]\qquad(11)$$

where $1+a+c+e=b+d$. This is implied in [19, II-24: $q\to 1$] and in one of Thomae's 120 transformations (cf. [2, §3.7: Equation 5]).

Consequently, we derive, by substitution, the expression

$$\Re(\mathrm{CC})=\mathcal{G}_\phi(a,b,c,d,e)\phi_3(a,b,c,d,e)+\mathcal{G}_\psi(a,b,c,d,e)\psi_3(a,b,c,d,e)\mathcal{F}(a,b,c,d,e),$$

where ϕ_3 and ψ_3 are quotients of trigonometric functions

$$\phi_3(a,b,c,d,e)=2\sin\pi\left[\begin{matrix}a-3c,\frac{1}{2}+2c,b+d-a-c,\frac{1}{2}+a-b-c,\frac{1}{2}+a-c-d,\frac{1}{2}+a-b+c-d,\frac{1}{2}+a-b-d+e\\c-e,2c,b-2c,d-2c,\frac{1}{2}+2a-b-2c-d\end{matrix}\right],$$

$$\psi_3(a,b,c,d,e)=\frac{1}{4}\sin\pi\left[\begin{matrix}3c-a,2a-2b-2c,2a-2c-2d,2a+2c-2b-2d,\frac{1}{2}+2c\\c-e,2c,b-2c,d-2c,\frac{1}{2}+a-2c-e,\frac{1}{2}+2a-b-2c-d\end{matrix}\right].$$

Recalling that the function $\Omega(z)$ is also symmetric with respect to c and e, we have

$$\Re(\text{EE}) = \mathcal{G}_\phi(a,b,c,d,e)\phi_3(a,b,e,d,c) + \mathcal{G}_\psi(a,b,c,d,e)\psi_3(a,b,e,d,c)\mathcal{F}(a,b,c,d,e).$$

2.4. Residue Sum over CE

For the residue sum over CE, we have

$$\Re(\text{CE}) = \sum_{n=0}^{\infty} \text{Res}_{z=2c+2e-2a-1-n}\{\Omega(z)\} = \frac{-2\pi}{\tan \pi(1+2a-2c-2e)}$$
$$\times \Gamma\begin{bmatrix} b+2c+2e-2a-1, 2c+d+2e-2a-1, 2c+2e-b-d-\frac{1}{2}, 4c+2e-2a-1, 2c+4e-2a-1 \\ 2c+2e-2b, 2c+2e-2d, 2b+2c+2d+2e-4a-1, c+2e-a, 2c+e-a, 3c+3e-2a-\frac{3}{2} \end{bmatrix}$$
$$\times\ {}_7F_6\begin{bmatrix} \frac{3}{2}+2a-3c-3e, 2+\frac{4a}{3}-2c-2e, 1+a-c-2e, 1+a-2c-e \\ 1+\frac{4a}{3}-2c-2e, 2+2a-4c-2e, 2+2a-2c-4e \end{bmatrix} \vdots$$
$$\vdots \begin{matrix} 1+2b-2c-2e, 1-2c+2d-2e, 2+4a-2b-2c-2d-2e \\ 2+2a-b-2c-2e, 2+2a-2c-d-2e, \frac{3}{2}+b-2c+d-2e \end{matrix} ;1\Bigg],$$

which can be restated, in view of Lemma 2, as

$$\Re(\text{CE}) = \frac{-2\pi}{\tan \pi(1+2a-2c-2e)}$$
$$\times \Gamma\begin{bmatrix} \frac{1}{2}, b+2c+2e-2a-1, 2+2a-b-2c-2e, 2c+d+2e-2a-1, 2+2a-2c-d-2e \\ 3c+3e-2a-\frac{3}{2}, \frac{5}{2}+2a-3c-3e, 1+b-c-e, 1+d-c-e, 2c+2e-2b, 2c+2e-2d \end{bmatrix}$$
$$\times \Gamma\begin{bmatrix} 2c+2e-b-d-\frac{1}{2}, \frac{3}{2}+b-2c+d-2e, 2c+4e-2a-1, 4c+2e-2a-1 \\ c+2e-a, 2c+e-a, 2b+2c+2d+2e-4a-1, \frac{3}{2}+2a-b-c-d-e \end{bmatrix}$$
$$\times\ {}_3F_2\begin{bmatrix} 1+2a-b-c-d-e, \frac{1}{2}+b-c-e, \frac{1}{2}-c+d-e \\ \frac{3}{2}+a-2c-e, \frac{3}{2}+a-c-2e \end{bmatrix};1\Bigg].$$

We get therefore a relatively shorter expression

$$\Re(\text{CE}) = \mathcal{G}_\psi(a,b,c,d,e)\psi_4(a,b,c,d,e)\mathcal{F}(a,b,c,d,e),$$

where ψ_4 is the quotient of trigonometric functions

$$\psi_4(a,b,c,d,e) = \frac{1}{4}\sin \pi\begin{bmatrix} 2b-2c-2e, 2d-2c-2e, 4a-2b-2c-2d-2e \\ 2a-2c-2e, 2a-b-2c-2e, 2a-2c-d-2e \end{bmatrix}$$
$$\times \sin\pi\begin{bmatrix} \frac{1}{2}+2a-2c-2e, \frac{1}{2}+2a-3c-3e \\ \frac{1}{2}+b-2c+d-2e, \frac{1}{2}+a-c-2e, \frac{1}{2}+a-2c-e \end{bmatrix}.$$

3. Main Theorem and Formulae

According to the linear equality (7), we obtain, by putting the six residue sums together, the following expression

$$\mathcal{H}(a,b,c,d,e) = \frac{\Re(\text{AA})}{2a} \times \Gamma\begin{bmatrix} 1+2a-2b, 1+2a-2d, 2b+2d-2a, 1+a-c, 1+a-e, \frac{1}{2}+c+e \\ b, d, \frac{1}{2}+2a-b-d, 2c, 2e, 1+2a-2c-2e \end{bmatrix}$$

$$=\frac{1}{2a}\Gamma\left[\begin{matrix}1+2a-2b,1+2a-2d,2b+2d-2a,1+a-c,1+a-e,\frac{1}{2}+c+e\\ b,d,\frac{1}{2}+2a-b-d,2c,2e,1+2a-2c-2e\end{matrix}\right]$$
$$\times\{\mathcal{G}_{\phi}(a,b,c,d,e)\Phi(a,b,c,d,e)+\mathcal{G}_{\psi}(a,b,c,d,e)\Psi(a,b,c,d,e)\mathcal{F}(a,b,c,d,e)\},$$

where Φ and Ψ are linear combinations of trigonometric quotients

$$\begin{aligned}\Phi(a,b,c,d,e):=&-\phi_1(a,b,c,d,e)-\phi_1(a,d,c,b,e)-\phi_2(a,b,c,d,e)\\&-\phi_3(a,b,c,d,e)-\phi_3(a,b,e,d,c),\\ \Psi(a,b,c,d,e):=&-\psi_1(a,b,c,d,e)-\psi_1(a,d,c,b,e)-\psi_2(a,b,c,d,e)\\&-\psi_3(a,b,c,d,e)-\psi_3(a,b,e,d,c)-\psi_4(a,b,c,d,e).\end{aligned}$$

We have therefore established the following remarkable evaluation theorem.

Theorem 3. *For the bilateral quadratic series $\mathcal{H}$ defined by (3), the following summation formula holds:*

$$\mathcal{H}(a,b,c,d,e)=\frac{1}{2a}\times\Gamma\left[\begin{matrix}1+2a-2b,1+2a-2d,2b+2d-2a,1+a-c,1+a-e,\frac{1}{2}+c+e\\ b,d,\frac{1}{2}+2a-b-d,2c,2e,1+2a-2c-2e\end{matrix}\right]$$
$$\times\left\{\mathcal{G}_{\phi}(a,b,c,d,e)\Phi(a,b,c,d,e)+\mathcal{G}_{\psi}(a,b,c,d,e)\Psi(a,b,c,d,e)\mathcal{F}(a,b,c,d,e)\right\}.$$

By making use of Mathematica commands, the two coefficients Φ and Ψ are miraculously simplified into the following elegant expressions:

$$\begin{aligned}\Phi(a,b,c,d,e)=&\frac{\cos\pi(a-b)\cos\pi(a-d)\cos\pi(a-b+c-d)\cos\pi(a-b-d+e)}{\sin(\pi b)\sin(\pi c)\sin(\pi d)\sin(\pi e)}\\&\times\frac{\sin(\pi a)\sin\pi(b+d-a)}{\cos\pi(b+d-2a)}\left\{1-\frac{\tan\pi(a-b)\tan\pi(a-d)\tan(\pi e)}{\tan(\pi a)\tan\pi(a-b-d)\cot(\pi c)}\right\},\\ \Psi(a,b,c,d,e)=&-1-\sin\pi\left[\begin{matrix}a-c,a-e,2a-2b,2a-2d,2a-2b-2d,\frac{1}{2}+c+e\\ b,d,2c,2e,2a-2c-2e,\frac{1}{2}+2a-b-d\end{matrix}\right].\end{aligned}$$

When one of the numerator parameters is a negative integer and one of the denominator parameters is a positive integer, the bilateral $\mathcal{H}$-series becomes terminating. In this case, several terminating quadratic ${}_7F_6$-series can be evaluated in closed forms by means of Theorem 3. We shall not reproduce them because they have been worked out by the modified Abel lemma on summation by parts (cf. [20, Corollaries 12, 13, 15 and 16]) and the inverse series relations (cf. [10, Equations 1.7 & 1.8] and [21, Equation 4.2b] and [22, Equations 4.1d, 4.2d & 5.1e]).

Instead, we shall derive, by making use of Theorem 3, five summation formulae when the $\mathcal{H}$-series is nonterminating. They will be presented in two subsections, respectively, for bilateral and unilateral series.

3.1. Summation Formulae for Bilateral Series

Firstly, let $1+a=d+e$ in Theorem 3. In this case, the balanced $\mathcal{F}$-series reduces to a ${}_2F_1$-series, which can be evaluated by the Gauss summation formula (cf. Bailey [2, §1.3])

$$\mathcal{F}(a,b,c,d,1+a-d)={}_2F_1\left[\begin{matrix}a-b-c,b+d-a-c-\frac{1}{2}\\ \frac{1}{2}-2c+d\end{matrix};1\right]$$
$$=\Gamma\left[\begin{matrix}\frac{1}{2}-2c+d\\ 1+a-b-c,\frac{1}{2}-a-c+b+d\end{matrix}\right].$$

Observe that $\mathcal{G}_\psi(a,b,c,d,1+a-d)=0$ because of the presence of $\Gamma(0)$ in the denominator. We have

$$\mathcal{H}(a,b,c,d,1+a-d)=\Gamma\begin{bmatrix}1+2a-2b,2b+2d-2a,1+a-c,\frac{3}{2}+a+c-d\\ b,\frac{1}{2}+2a-b-d,2c,2d-2c-1\end{bmatrix}$$
$$\times\Psi(a,b,c,d,1+a-d)\frac{\mathcal{F}(a,b,c,d,1+a-d)}{2a(1+2a-2d)}\mathcal{G}_\psi(a,b,c,d,1+a-d).$$

By simplifying the product

$$\frac{\mathcal{F}(a,b,c,d,1+a-d)}{2a(1+2a-2d)}\mathcal{G}_\psi(a,b,c,d,1+a-d)$$
$$=\frac{1}{a(a-b-c)(1+2a-2d)(2b+2d-2a-2c-1)},$$

we find the following closed formula.

Proposition 4. *For the bilateral quadratic series $\mathcal{H}$ defined by (3), we have the following closed formula*:

$$\mathcal{H}(a,b,c,d,1+a-d)=\Gamma\begin{bmatrix}2a-2b,2b+2d-2a,a-c,\frac{1}{2}+a+c-d\\ b,\frac{1}{2}+2a-b-d,2c,2d-2c\end{bmatrix}$$
$$\times\frac{(a-b)(a-c)(1+2a+2c-2d)(1+2c-2d)}{a(b+c-a)(1+2a-2d)(1+2a+2c-2b-2d)}$$
$$\times\left\{1-\sin\pi\begin{bmatrix}\frac{1}{2}+a+c-d,2a-2b,a-c,2a-2b-2d\\ \frac{1}{2}+2a-b-d,b,2c,2c-2d\end{bmatrix}\right\}.$$

Analogously, let $a+e=\frac{1}{2}+b+d$ in Theorem 3. In this case, the balanced $\mathcal{F}$-series reduces again to a ${}_2F_1$-series which can be evaluated as follows:

$$\mathcal{F}(a,b,c,d,\tfrac{1}{2}-a+b+d)={}_2F_1\begin{bmatrix}a-b-c,a-c-d\\1+2a-2c-b-d\end{bmatrix};1\Bigg]$$
$$=\Gamma\begin{bmatrix}1+2a-2c-b-d\\1+a-b-c,1+a-c-d\end{bmatrix}.$$

Since $\Phi(a,b,c,d,\frac{1}{2}-a+b+d)=0$, the corresponding quadratic series is evaluated in the following proposition.

Proposition 5. *For the bilateral quadratic series $\mathcal{H}$ defined by (3), we have the following closed formula*:

$$\mathcal{H}(a,b,c,d,\tfrac{1}{2}-a+b+d)=\Gamma\begin{bmatrix}2a-2b,a-c,2a-2d,b+c+d-a\\ b,2c,d,4a-2b-2c-2d\end{bmatrix}$$
$$\times\frac{(a-b)(a-c)(a-d)(b+c+d-a)}{a(a-b-c)(a-c-d)(a-b-d)}$$
$$\times\left\{1+\sin\pi\begin{bmatrix}2a-2b,a-c,2a-2d,a-b-c-d\\ b,2c,d,4a-2b-2c-2d\end{bmatrix}\right\}.$$

Finally, when $b\to\frac{a}{3}+c$ and $d\to\frac{a}{3}+e$, the corresponding $\mathcal{F}$-series in Theorem 3 can be evaluated by Dixon's summation formula (cf. Bailey [2, §3.1]):

$$\mathcal{F}(a,\tfrac{a}{3}+c,c,\tfrac{a}{3}+e,e)=\Gamma\begin{bmatrix}\frac{1}{2},\frac{3}{2}+\frac{2a}{3}-c-e,\frac{3}{2}+a-2c-e,\frac{3}{2}+a-c-2e\\ 1+\frac{a}{3}-c,1+\frac{a}{3}-e,2+\frac{4a}{3}-2c-2e,1+\frac{2a}{3}-c-e\end{bmatrix}.$$

By means of the Legendre duplication formula, it can be verified that

$$\frac{\mathcal{G}_\psi(a,\frac{a}{3}+c,c,\frac{a}{3}+e,e)}{\mathcal{G}_\phi(a,\frac{a}{3}+c,c,\frac{a}{3}+e,e)}\times\mathcal{F}\left(a,\frac{a}{3}+c,c,\frac{a}{3}+e,e\right)=\frac{1}{2}.$$

Therefore, we have the following expression

$$\mathcal{H}(a,\frac{a}{3}+c,c,\frac{a}{3}+e,e)=\Gamma\begin{bmatrix}1+\frac{4a}{3}-2c,1+\frac{4a}{3}-2e,2c+2e-\frac{2a}{3},1+a-c,1+a-e,\frac{1}{2}+c+e\\ \frac{a}{3}+c,\frac{a}{3}+e,\frac{1}{2}+\frac{4a}{3}-c-e,2c,2e,1+2a-2c-2e\end{bmatrix}$$
$$\times\frac{\mathcal{G}_\phi(a,\frac{a}{3}+c,c,\frac{a}{3}+e,e)}{4a}\left\{2\Phi(a,\frac{a}{3}+c,c,\frac{a}{3}+e,e)+\Psi(a,\frac{a}{3}+c,c,\frac{a}{3}+e,e)\right\}$$

where $\mathcal{G}_\phi$ is given explicitly by

$$\mathcal{G}_\phi\left(a,\frac{a}{3}+c,c,\frac{a}{3}+e,e\right)=\Gamma\begin{bmatrix}1+\frac{4a}{3}-2c-2e,\frac{1}{2}+\frac{a}{3}-c,\frac{1}{2}+\frac{a}{3}-c,\frac{1}{2}+\frac{a}{3}-e,\frac{1}{2}+\frac{a}{3}-e\\ 1+\frac{2a}{3}-2c,1+\frac{2a}{3}-2e,1+\frac{2a}{3}-c-e,1+\frac{2a}{3}-c-e\end{bmatrix}.$$

By executing Mathematica commands, we find a slightly reduced expression

$$2\Phi(a,\frac{a}{3}+c,c,\frac{a}{3}+e,e)+\Psi(a,\frac{a}{3}+c,c,\frac{a}{3}+e,e)=\frac{\cos\,\pi(\frac{2a}{3}-c-e)}{\cos\,\pi(\frac{4a}{3}-c-e)}\Theta(a,c,e)$$

where the trigonometric expression $\Theta(a,c,e)$ is determined by

$$\begin{aligned}&\Theta(a,c,e)\\&=\frac{\sin\left(\frac{4\pi a}{3}\right)\cos\pi\left(\frac{a}{3}-c\right)\cos\pi\left(\frac{a}{3}-e\right)}{\sin\pi\left(\frac{a}{3}+c\right)\sin\pi\left(\frac{a}{3}+e\right)\sin(2\pi c)\sin(2\pi e)\sin\pi(2a-2c-2e)}\\&\times\left\{\begin{matrix}\left(1+2\cos\pi\left(\frac{2a}{3}-2c\right)\right)\left(\sin\pi\left(\frac{a}{3}+c\right)\sin\pi(a-c)-\sin^2\pi(a-c-2e)\right)\\ -2\sin\pi\left(\frac{a}{3}-c\right)\left(1+2\cos\pi(a-c)\cos\pi\left(\frac{a}{3}+c\right)\right)\sin\pi(a-c-2e)\end{matrix}\right\}.\end{aligned}\tag{12}$$

We remark that the above trigonometric expression in the braces '{· · · }' is symmetric with respect to 'c and e', even though it does not look like so.

Hence, we have established, after some routine simplifications of the Γ-function quotient, the following evaluation.

Proposition 6. *For the bilateral quadratic series $\mathcal{H}$ defined by (3), we have the following closed formula*

$$\mathcal{H}(a,\frac{a}{3}+c,c,\frac{a}{3}+e,e)=\frac{\sqrt{\pi}}{4a}\Gamma(\star)\Theta(a,c,e),$$

where Θ is defined by (12) and $\Gamma(\star)$ by

$$\Gamma(\star)=\Gamma\begin{bmatrix}1+a-c,1+a-e,1+\frac{4a}{3}-2c,1+\frac{4a}{3}-2e,\frac{1}{2}+\frac{a}{3}-c,\frac{1}{2}+\frac{a}{3}-e\\ 1+2a-2c-2e,2c,2e,\frac{a}{3}+c,\frac{a}{3}+e,1+\frac{a}{3}-c,1+\frac{a}{3}-e\end{bmatrix}$$
$$\times\Gamma\begin{bmatrix}\frac{1}{2}+c+e,2c+2e-\frac{2a}{3},\frac{1}{2}-\frac{4a}{3}+c+e\\ 1+\frac{2a}{3}-c-e,\frac{1}{2}-\frac{2a}{3}+c+e\end{bmatrix}.$$

3.2. Summation Formulae for Unilateral Series

When $d=a$, it is routine to check that

$$\Phi(a,b,c,a,e)=\frac{\cos\,\pi(b-c)\cos\,\pi(b-e)}{\sin\,(\pi c)\sin\,(\pi e)}\quad\text{and}\quad\Psi(a,b,c,a,e)=-1.$$

The corresponding formula in Theorem 3 reads as

$$\begin{aligned}\mathcal{H}(a,b,c,a,e)=\frac{1}{2}\Gamma\left[\begin{matrix}1+2a-2b,2b,1+a-c,1+a-e,\frac{1}{2}+c+e\\1+a,b,\frac{1}{2}+a-b,2c,2e,1+2a-2c-2e\end{matrix}\right]\\\times\left\{\mathcal{G}_{\phi}(a,b,c,a,e)\Phi(a,b,c,a,e)-\mathcal{G}_{\psi}(a,b,c,a,e)\mathcal{F}(a,b,c,a,e)\right\}\end{aligned}$$

which can be simplified into the following formula.

Proposition 7. *There holds the evaluation for the unilateral quadratic series*:

$$\begin{aligned}\mathcal{H}(a,b,c,a,e)=\Gamma\left[\begin{matrix}\frac{1}{2},1+a-b,1+a-c,1+a-e,1+a-b-c-e,\frac{1}{2}+b,\frac{1}{2}+c+e,\frac{1}{2}+b-c-e\\1+a,1+a-b-c,1+a-b-e,1+a-c-e,\frac{1}{2}+c,\frac{1}{2}+e,\frac{1}{2}+b-c,\frac{1}{2}+b-e\end{matrix}\right]\\\times\left\{1-\mathcal{F}(a,b,c,a,e)\times\Gamma\left[\begin{matrix}\frac{1}{2}+b-c,\frac{1}{2}+b-e,1+a-b-c,1+a-b-e\\c,e,\frac{3}{2}+a-2c-e,\frac{3}{2}+a-c-2e\end{matrix}\right]\right\}.\end{aligned}$$

The last formula is equivalent to the beautiful one displayed in Lemma 2 which can be justified by reformulating the above expression inside the braces "$\{\cdots\}$" by

$$\begin{aligned}&\mathcal{F}(a,b,c,d,e)\\&=\Gamma\left[\begin{matrix}c+d-a,d+e-a,\frac{3}{2}+a-2c-e,\frac{3}{2}+a-c-2e\\1+a-b-c,1+a-b-e,\frac{1}{2}-a-c+b+d,\frac{1}{2}-a-e+b+d\end{matrix}\right]\\&-\ {}_3F_2\left[\begin{matrix}c+d-a,d+e-a,\frac{1}{2}-c+d-e\\1-b+d,\frac{1}{2}-2a+b+2d\end{matrix};1\right]\\&\times\Gamma\left[\begin{matrix}c+d-a,d+e-a,\frac{3}{2}+a-2c-e,\frac{3}{2}+a-c-2e\\1+2a-b-c-d-e,1-b+d,\frac{1}{2}+b-c-e,\frac{1}{2}-2a+b+2d\end{matrix}\right]\end{aligned}\tag{13}$$

The reciprocal relation in (13) can be, in turn, derived by applying the two term relation due to Thomae (1879, cf. Bailey [2, §3.2: Equation 2]) and then the non-terminating Pfaff–Saalchützian summation formula (11).

Alternatively, for $e=a$, we have

$$\begin{aligned}\Phi(a,b,c,d,a)=\frac{\cos\,\pi(a-b-c)\cos\,\pi(a+c-b-d)}{\sin\,(2\pi c)\sin\,(\pi d)}\\\times\{1-\frac{\cos\,\pi(a-b+c)\sin\,\pi(2a-b-2d)}{\cos\,\pi(a-b-c)\sin\,(\pi b)}\},\\\Psi(a,b,c,d,a)=-1.\end{aligned}$$

The corresponding formula in Theorem 3 can be restated as

$$\begin{aligned}\mathcal{H}(a,b,c,d,a)=\Gamma\left[\begin{matrix}1+2a-2b,1+2a-2d,2b+2d-2a,1+a-c,\frac{1}{2}+a+c\\b,d,\frac{1}{2}+2a-b-d,1+2a,2c,1-2c\end{matrix}\right]\\\times\left\{\mathcal{G}_{\phi}(a,b,c,d,a)\Phi(a,b,c,d,a)-\mathcal{G}_{\psi}(a,b,c,d,a)\mathcal{F}(a,b,c,d,a)\right\}.\end{aligned}$$

By canceling the common factors, we get an explicit formula.

Proposition 8. *There holds the evaluation for the unilateral quadratic series*:

$$\mathcal{H}(a,b,c,d,a)=\frac{\cos\pi(a-b-c)}{\cos\pi(a-b+c)}\Gamma\begin{bmatrix}1+a-c,\frac{1}{2}+a+c,\frac{1}{2}+d-a-c\\ \frac{1}{2}+a-b+c,\frac{1}{2}+b+d-a-c\end{bmatrix}$$
$$\times\Gamma\begin{bmatrix}1+a-b-c-d,1+2a-2b,1+2a-2d,2b+2d-2a\\ 1+2a,1+a-b-c,1+a-c-d,b,1-b\end{bmatrix}$$
$$\times\left\{1-\frac{\cos\pi(a-b+c)\sin\pi(2a-b-2d)}{\cos\pi(a-b-c)\sin(\pi b)}-\frac{\sin(2\pi c)\mathcal{F}(a,b,c,d,a)}{\cos\pi(a-b-c)}\right.$$
$$\left.\times\Gamma\begin{bmatrix}1-b,1+a-b-c,1+a-c-d,\frac{1}{2}-a-c+b+d\\ d,\frac{1}{2}+2a-b-d,\frac{3}{2}-a-c,\frac{3}{2}-2c\end{bmatrix}\right\}.$$

The above formula is equivalent to another formula displayed in Lemma 1, which can be restated in the following manner:

$$\mathcal{H}(a,b,c,d,a)=\frac{\sin(2\pi c)}{\cos\pi(a+c-b-d)}\Gamma\begin{bmatrix}\frac{1}{2}+a+c,\frac{1}{2}+d-a-c\\ \frac{1}{2}+a-b+c,\frac{1}{2}+b+d-a-c\end{bmatrix}$$
$$\times\Gamma\begin{bmatrix}1+a-c,1+a-b-c-d,1+2a-2b,1+2a-2d,2b+2d-2a\\ 1+2a,1+a-b-c,1+a-c-d,b,1-b\end{bmatrix}$$
$$\times\left\{\frac{\sin\pi(2b+2d-2a)\cos\pi(a+c-d)}{\sin(\pi b)\sin(2\pi c)}\right.$$
$$+\Gamma\begin{bmatrix}1-b,1+a-b-c,1+a-c-d,\frac{1}{2}+a-b+c\\ d,\frac{1}{2}+d-a-c,\frac{3}{2}+2a-b-2d,\frac{3}{2}+2a-2b-d\end{bmatrix} \tag{14}$$
$$\left.\times\ {}_3F_2\begin{bmatrix}1+a-b-c-d,\frac{1}{2}+2a-b-d,\frac{1}{2}+a-b+c-d\\ \frac{3}{2}+2a-2b-d,\frac{3}{2}+2a-b-2d\end{bmatrix};1\right\}.$$

In fact, by applying (11) to the ${}_3F_2$-series on the right hand side of (13), we get another reciprocal relation for $\mathcal{F}(a,b,c,d,e)$:

$$\mathcal{F}(a,b,c,d,e)$$
$$=\Gamma\begin{bmatrix}\frac{1}{2}+a+c-b-d,\frac{1}{2}+a+e-b-d,\frac{3}{2}+a-2c-e,\frac{3}{2}+a-c-2e\\ 1+a-b-c,1+a-b-e,1+a-c-d,1+a-d-e\end{bmatrix}$$
$$-\ {}_3F_2\begin{bmatrix}1+2a-b-c-d-e,\frac{1}{2}+a+c-b-d,\frac{1}{2}+a+e-b-d\\ \frac{3}{2}+2a-2b-d,\frac{3}{2}+2a-b-2d\end{bmatrix};1 \tag{15}$$
$$\times\Gamma\begin{bmatrix}\frac{1}{2}+a+c-b-d,\frac{1}{2}+a+e-b-d,\frac{3}{2}+a-2c-e,\frac{3}{2}+a-c-2e\\ \frac{1}{2}+b-c-e,\frac{1}{2}+d-c-e,\frac{3}{2}+2a-2b-d,\frac{3}{2}+2a-b-2d\end{bmatrix}.$$

Then Rahman's formula (14) follows by rewriting the ${}_3F_2$-series in Proposition 8 by (15) and then factorizing the trigonometric expression

$$1-\frac{\sin\pi(2a-b-2d)\cos\pi(a-b+c)}{\sin(\pi b)\cos\pi(a-b-c)}-\frac{\sin(2\pi c)\sin(\pi d)}{\cos\pi(a-b-c)\cos\pi(a+c-b-d)}$$
$$=\frac{\sin\pi(2b+2d-2a)\cos\pi(a-b+c)\cos\pi(a+c-d)}{\sin(\pi b)\cos\pi(a-b-c)\cos\pi(a+c-b-d)}.$$

4. Conclusion and Further Comment

By making use of the Cauchy residue method (the contour integration approach), we have successfully evaluated a difficult bilateral quadratic $\mathcal{H}$-series. This will be shown to be a significant contribution to the theory of hypergeometric series and special functions. There exist terminating cubic and quartic series in the mathematical literature (for example, [10,18,20]), it would be equally important to establish their bilaterally nonterminating counterparts as we have done for the quadratic one. The interested reader is enthusiastically encouraged to make further exploration.

Author Contributions: Formal analysis, computation and Writing, X.C.; Methodology, Review and supervision, W.C.

Funding: This research received no external funding.

Acknowledgments: The authors express their reciprocal appreciations for efficient collaborations.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Rainville, E.D. *Special Functions*; The Macmillan Company Press: New York, NY, USA, 1960.
2. Bailey, W.N. *Generalized Hypergeometric Series*; Cambridge Tracts in Mathematics and Mathematical Physics; Cambridge University Press: Cambridge, UK, 1935.
3. Slater, L.J. *Generalized Hypergeometric Functions*; Cambridge University Press: Cambridge, UK, 1966.
4. Brychkov, Y.A. *Handbook of Special Functions: Derivatives, Integrals, Series and Other Formulas*; Chapman and Hall/CRC: Boca Raton, FL, USA, 2008.
5. Chu, W. Inversion techniques and combinatorial identities: A quick introduction to hypergeometric evaluations. *Math. Appl.* **1994**, *283*, 31–57.
6. Chu, W. Inversion techniques and combinatorial identities: Balanced hypergeometric series. *Rocky Mt. J. Math.* **2002**, *32*, 561–587.
7. Chu, W. Abel's method on summation by parts and hypergeometric series. *J. Difference Equ. Appl.* **2006**, *12*, 783–798, https://doi.org/10.1080/10236190600704096.
8. Ekhad, S.B. Forty strange computer–discovered and computer–proved (of course) hypergeometric series evaluations. Available online: http://www.math.rutgers.edu/~zeilberg/ekhad/ekhad.html (accessed on 12 October 2004).
9. Gessel, I.M. Finding Identities with the WZ Method. *J. Symb. Comput.* **1995**, *20*, 537–566, https://doi.org/10.1006/jsco.1995.1064.
10. Gessel, I.; Stanton, D. Strange Evaluations of Hypergeometric Series. *SIAM J. Math. Anal.* **1982**, *13*, 295–308, https://doi.org/10.1137/0513021.
11. Stanton, D. A Hypergeometric Hierarchy for the Andrews Evaluations. *Ramanujan J.* **1998**, *2*, 499–509, https://doi.org/10.1023/a:1009784911098.
12. Dougall, M.A. On Vandermonde's theorem and some more general expansion. *Proc. Edin. Math. Soc.* **1907**, *25*, 114–132.
13. Jackson, M. A Generalization of the Theorems of Watson and Whipple on the Sum of the Series ${}_3F_2$. *J. Lond. Math. Soc.* **1949**, *s1-24*, 238–240, https://doi.org/10.1112/jlms/s1-24.3.238.
14. Jackson, M. A Note on the Sum of a Particular Well-Poised ${}_6H_6$ with Argument–1. *J. Lond. Math. Soc.* **1952**, *1*, 124–126, https://doi.org/10.1112/jlms/s1-27.1.124.
15. Chu, W. Asymptotic method for Dougall's bilateral hypergeometric sums. *Bull. Sci. Math.* **2007**, *131*, 457–468, https://doi.org/10.1016/j.bulsci.2006.09.002.
16. Chu, W.; Wang, X.; Zheng, D. Application of the residue theorem to bilateral hypergeometric series. *Le Matematiche* **2007**, *62*, 127-146.
17. Titchmarsh, E.C. *The Theory of Functions*, 2nd ed.; Oxford University Press: Oxford, UK, 1952.
18. Rahman, M. Some Quadratic and Cubic Summation Formulas for Basic Hypergeometric Series. *Can. J. Math.* **1993**, *45*, 394–411, https://doi.org/10.4153/cjm-1993-020-8.

19. Gasper, G.; Rahman, M. *Basic Hypergeometric Series*, 2nd ed.; Encyclopedia of Mathematics and Its Applications; Cambridge University Press: Cambridge, MA, USA, 2004; Volume 96.
20. Chu, W.; Wang, X. The modified Abel lemma on summation by parts and terminating hypergeometric series identities. *Integral Transform. Spéc. Funct.* **2009**, *20*, 93–118, https://doi.org/10.1080/10652460802499828.
21. Chu, W. Inversion techniques and combinatorial identities. *Boll. Un. Mat. Ital. B* **1993**, *7*, 737–760.
22. Chu, W. Inversion techniques and combinatorial identities: A unified treatment for the $_7F_6$-series identities. *Collect. Math.* **1994**, *45*, 13–43.